EE8725 HW 5

Solve the following optimization problem using the interior point algorithm:

$$\min f(x_1, x_2) = 0.25x_1^2 + x_2^2$$
$$h(x_1, x_2) = x_2 - 0.05x_1^2 - 0.5x_1 + 2 \le 0$$

See attached pages for theory of interior point algorithm

8.1 APPENDIX 8A: Interior Point Method

The interior point method converts the inequality constraints to equality constraints using the same technique of adding a slack variable to each constraint. Then a "penalty function" is added to the objective function so that the equations become:

Minimize: $f_\mu = f(\underline{x}) - \mu^k \sum_{i=1}^{N_{ineq}} \ln(s_i)$, where N_{ineq} is the number if inequality constraints.

Subject to: $g(\underline{x}) = 0$

and $h(\underline{x}) + s = 0$

where $\mu^k \geq 0$ is the *barrier parameter* which is forced to decrease toward zero as the algorithm iterates to a solution (k is the iteration counter). The Lagrangian now looks like this:

$$L_\mu = f(x) - \mu^k \sum_{i=1}^{N_{ineq}} \ln(s_i) + \lambda^T g(x) + \gamma^T (h(x) + s)$$

And the Gradient of this Lagrangian becomes:

$$\nabla_x L_\mu = \nabla_x f(x) + \nabla_x g(x)\lambda + \nabla_x h(x)\gamma = 0$$

$$\nabla_\lambda L_\mu = g(x) = 0$$

$$\nabla_\mu L_\mu = h(x) + s = 0$$

$$\nabla_s L_\mu = -\mu^k S^{-1} e + \gamma = 0$$

Where $S = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & s_3 & \\ & & & \ddots \end{bmatrix}$ and $e = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$

The algorithm at iteration k requires the solution of the following set of equations:

$$\begin{bmatrix} \nabla_x^2 L_\mu & \nabla_x g(x) & \nabla_x h(x) & 0 \\ \nabla_x g(x)^T & 0 & 0 & 0 \\ \nabla_x h(x)^T & 0 & 0 & I \\ 0 & 0 & I & \nabla_s^2 L_\mu \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta \gamma \\ \Delta s \end{bmatrix} = \begin{bmatrix} \nabla_x L_\mu \\ \nabla_\lambda L_\mu \\ \nabla_\gamma L_\mu \\ \nabla_s L_\mu \end{bmatrix}$$

Solution of the Optimization Problem of Appendix 3A using Interior Point Algorithm[1]

The optimization problem of Appendix 3A with one equality constraint and one inequality constraint is as given below:

$$\min f(x_1, x_2) = 0.25x_1^2 + x_2^2$$
$$g(x_1, x_2) = 5 - x_1 - x_2 = 0$$
$$h(x_1, x_2) = x_1 + 0.2x_2 - 3 \le 0$$

To make this a more interesting problem we shall drop the equality constraint and reverse the direction of the inequality constraint, we then solve the problem:

$$\min f(x_1, x_2) = 0.25x_1^2 + x_2^2$$
$$h(x_1, x_2) = -x_1 - 0.2x_2 + 3 \le 0$$

First we add a slack variable to the inequality constraint and the log barrier function to the objective:

$$\min f(x_1, x_2) = 0.25x_1^2 + x_2^2 - \mu^k \ln(s)$$
$$h(x_1, x_2) = -x_1 - 0.2x_2 + 3 + s = 0$$

The Lagrangian function is:

$$L(x_1, x_2, \gamma, s) = 0.25x_1^2 + x_2^2 - \mu^k \ln(s) + \gamma\left(-x_1 - 0.2x_2 + 3 + s\right)$$

And the minimum is found at the point $\left(x_1^*, x_2^*, \gamma^*, s^*\right)$ where:

$$\nabla_{x_1} L_\mu = 0.5x_1^* - \gamma^* = 0$$

[1] This solution was done for a class homework problem by Volker Landenberger

$$\nabla_{x_2} L_\mu = 2x_2^* - 0.2\gamma^* = 0$$

$$\nabla_\lambda L_\mu = -x_1^* - 0.2x_2^* + 3 + s^* = 0$$

$$\nabla_s L_\mu = -\frac{\mu^k}{s^*} + \gamma^* = 0$$

Starting at $x_1 = 10$, $x_2 = 10$, and $\mu = 20$ this solution of these equations with a reduction of μ by one half each iteration converges to the solution:

$x_1 = 2.9704$
$x_2 = 0.1485$
$\gamma = 1.4852$
$\mu = 0.0002$
$s = 0.0001$